Math 564: Real analysis and measure theory
Lecture 23

Lebesgue differentiation theorem. For any $f \in Loc^1(\mathbb{R}^d, \lambda)$,
$$\lim_{r\to 0} \frac{1}{\lambda(B_r(x))} \int_{B_r(x)} f \, d\lambda = f(x) \quad \text{for } \lambda\text{-a.e. } x \in \mathbb{R}^d.$$
Equivalently, for any loc. finite Borel measure μ on $\mathbb{R}^d$ with $\mu \ll \lambda$, we have
$$\frac{d\mu}{d\lambda}(x) = \lim_{r\to 0} \frac{\mu(B_r(x))}{\lambda(B_r(x))} \quad \text{for } \lambda\text{-a.e. } x \in \mathbb{R}^d.$$

We will prove this after a definition and lemmas. Let $A_r f := \frac{1}{\lambda(B_r(x))} \int_{B_r(x)} f \, d\lambda$ and call A_r the **averaging operator at radius r.** We want to prove:
$$\lim_{r\to 0} A_r f = f \quad \text{a.e.}$$

Local-global bridge lemma. Let $f \in L^1(\mathbb{R}^d, \lambda)$. For each $r > 0$, we have:

(a) $\int f \, d\lambda = \int A_r f \, d\lambda$.

(b) $\|A_r f\|_1 \le \|f\|_1$, i.e. A_r is an L^1-contraction.

Proof. HW.

Lemma. If $g \in Loc^1(\mathbb{R}^d, \lambda)$ is continuous, then $\lim_{r\to 0} A_r g = g$ everywhere.

Proof. $|A_r g(x) - g(x)| = \frac{1}{\lambda(B_r(x))} \left| \int (g(y) - g(x)) \, d\lambda(y) \right| \le \frac{1}{\lambda(B_r(x))} \int_{B_r(x)} |g(y) - g(x)| \, d\lambda(y) \le \sup_{y \in B_r(x)} |g(y) - g(x)| \xrightarrow[r\to 0]{} 0$ by continuity. $\square$

Proof of Lebesgue differentiation. It is enough to prove that the statement holds for $\mathbb{1}_{B_n(0)} \cdot f$ for all $n \in \mathbb{N}^+$ since ctbl union of null sets is null and for each $x \in B_n(0)$, the small ball $B_r(x) \subseteq B_n(0)$ by openness for all sufficiently small r. Thus, we fix $n \in \mathbb{N}$, so replacing f with $\mathbb{1}_{B_n(0)} \cdot f$, we may assume f is λ-integrable and forget $B_n(0)$.

We aim to prove that $A^* f := \limsup_{r\to 0} A_r f = f$ a.e., since the argument for $\liminf_{r\to 0}$ would be analogous.

Notation. For a function $h: X \to \mathbb{R}$ and $\alpha \in \mathbb{R}$, put $\{h > \alpha\} := \{x \in X : h(x) > \alpha\}$.

To show that $\{|A^* f - f| > 0\}$ is null, it is enough to show that $\{|A^* f - f| > \alpha\}$ is null for all $\alpha > 0$ because $\{|A^* f - f| > 0\} = \bigcup_{n \in \mathbb{N}^+} \{|A^* f - f| > \frac{1}{n}\}$. So we fix $\alpha > 0$. Letting $g \in L^1(\mathbb{R}^d, \lambda)$ be a continuous function, we see that

$$|A^* f - f| = |A^* f - A^* g + A^* g - g + g - f| \leq |A^* f - A^* g| + \overset{0}{\overset{\shortparallel}{|A^* g - g|}} + |g - f|$$
$$= |A^*(f-g)| + |g - f| \leq A^* |f-g| + |g - f|.$$

Therefore, $\{|A^* f - f| > \alpha\} \subseteq \underset{(b)}{\{A^* |f-g| > \alpha/2\}} \cup \underset{(a)}{\{|f-g| > \alpha/2\}}$, so it is enough to show that the last two sets would have arbitrarily small measure for an appropriate choice of g. Because continuous functions are dense in $L^1(\mathbb{R}^d, \lambda)$, we can make $\|f-g\|_1$ arbitrarily small, so it would suffice to show that each of these two sets has measure $\leq$ constant $\cdot \|f-g\|_1$ where the constant doesn't depend on g.

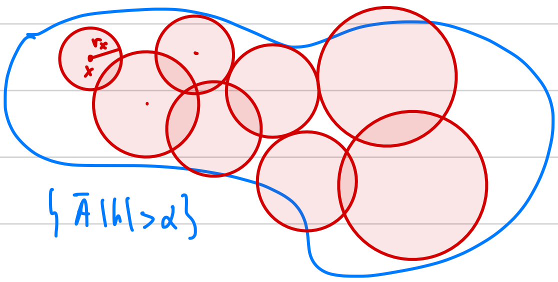
(a) By Chebyshev's inequality, $\frac{\alpha}{2} \cdot \lambda(\{|f-g| > \alpha/2\}) \leq \|f-g\|_1$, so $\lambda(\{|f-g| > \alpha\}) \leq \frac{2}{\alpha} \cdot \|f-g\|_1 \to 0$ as $g \to_{L^1} f$.

(b) We'd like to show again that $\lambda(\{A^* |f-g| > \alpha/2\}) \leq C \cdot \|f-g\|_1$ for some constant C, and this exactly what the following theorem says, so $\lambda(\{A^* |f-g| > \alpha/2\}) \leq C \cdot \|f-g\|_1 \to 0$ as $g \to_{L^1} f$. $\square$

Hardy-Littlewood Maximal Theorem. Let $h \in L^1(\mathbb{R}^d, \lambda)$ and $\alpha > 0$. Then
$$\lambda(\{A^* |h| > \alpha\}) \leq \frac{3^d}{\alpha} \|h\|_1.$$
In fact, we have $\lambda(\{\bar{A}|h| > \alpha\}) \leq 3^d/\alpha \, \|h\|_1$, where $\bar{A}|h| := \sup_{r \leq 1} A_r |h| \geq A^* |h|$ is the *Hardy-Littlewood maximal function*.

Proof. Note that for each $x \in \mathbb{R}^d$, we have $x \in \{\bar{A}|h| > \alpha\} \iff \exists\, r_x \in (0, 1]$ such that

$A_{r_x}|h| > \alpha$, i.e. $\frac{1}{\lambda(B_{r_x}(x))} \cdot \int_{B_{r_x}(x)} |h| \, d\lambda > \alpha$, i.e. $\lambda(B_{r_x}(x)) < \frac{1}{\alpha} \cdot \int_{B_{r_x}(x)} |h| \, d\lambda$.



It would be enough to get a ctbl subfamily of these balls $B_{r_x}(x)$ so that they are disjoint and cover a constant fraction (say half) of $\{\bar{A}|h| > \alpha\}$. This is exactly the content of the Vitali covering lemma below. Granted this lemma, we finish the proof as follows. Fix any $a > 0$ below $\lambda(\{\bar{A}|h| > \alpha\})$, and get a finite disjoint subcollection $\mathcal{C}_0 \subseteq \{B_{r_x}(x) : x \in \{\bar{A}|h| > \alpha\}\}$ with $\lambda(\bigsqcup_{B \in \mathcal{C}_0} B) \geq \frac{1}{3^d} \cdot a$.

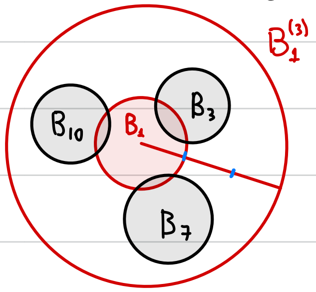
Then $\|h\|_1 \geq \int_{\bigcup_{B \in \mathcal{C}_0} B} |h| \, d\lambda = \sum_{B \in \mathcal{C}_0} \int_B |h| \, d\lambda > \alpha \cdot \sum_{B \in \mathcal{C}_0} \lambda(B) = \alpha \lambda(\bigsqcup_{B \in \mathcal{C}_0} B) \geq \frac{\alpha}{3^d} \cdot a \xrightarrow[a \to \lambda(\{\bar{A}|h| > \alpha\})]{} \frac{\alpha}{3^d} \lambda(\{\bar{A}|h| > \alpha\})$,

so $\lambda(\{\bar{A}|h| > \alpha\}) \leq \frac{3^d}{\alpha} \|h\|_1$. $\square$

Vitali Covering Lemma. Let $A \subseteq \mathbb{R}^d$ be any λ-measurable set of positive measure and let $\mathcal{C}$ be a family of balls that cover A. Then for each $0 < a < \lambda(A)$, there is a finite disjoint subcollection $\mathcal{C}_0 \subseteq \mathcal{C}$ such that

$$\lambda\Big(\bigsqcup_{B \in \mathcal{C}_0} B\Big) \geq \frac{1}{3^d} \cdot a.$$

Proof. Fix $a < \lambda(A)$ and by regularity get a compact $K \subseteq A$ with $\lambda(K) > a$. Then $\mathcal{C}$ is still a cover of K hence there is a finite subcover $\{B_1, B_2, \dots, B_n\} \subseteq \mathcal{C}$ of K. Order these balls by decreasing radii $\text{radius}(B_1) \geq \text{radius}(B_2) \geq \dots$ Put $B_{n_1} := B_1$ into $\mathcal{C}_0$. Delete the balls that intersect B_{n_1} and let B_{n_2} be a largest radius ball among the remaining balls. Delete the balls that intersect B_{n_2}, and let B_{n_3} be a largest radius ball among what remains...



Let $n_k := \min\{i \leq n : B_i \cap \bigcup_{j<k} B_{n_j} = \emptyset\}$. For a ball B, denote by $B^{(3)}$ the ball with the same center as B by with 3 times the radius of B. After this algorithm finishes, we have obtained a disjoint collection

$\mathcal{C}_0 := \{B_{n_1}, B_{n_2}, \dots, B_{n_\ell}\}$ such that $\bigcup_{i \le \ell} B_{n_i}^{(3)} \supseteq \bigcup_{j \le n} B_j \supseteq K$ because $B_{n_i}^{(3)}$ contains all B_j for $j \ge n_i$ which intersect B_{n_i}.

Thus, $\lambda\left(\bigsqcup_{i \le \ell} B_{n_i}\right) = \sum_{i \le \ell} \lambda(B_{n_i}) = \sum_{i \le \ell} \frac{1}{3^d} \lambda(B_{n_i}^{(3)}) \ge \frac{1}{3^d} \lambda\left(\bigcup_{i \le \ell} B_{n_i}^{(3)}\right) \ge \frac{1}{3^d} \cdot \lambda(K) \ge \frac{1}{3^d} \cdot a.$ $\square$

Technical Strengthening of Lebesgue differentiation thm. For each $f \in \mathrm{Loc}^1(\mathbb{R}^d, \lambda)$,

$$\lim_{r \to 0} \frac{1}{\lambda(B_r(x))} \int_{B_r(x)} |f(y) - f(x)| \, d\lambda(y) = 0 \quad \text{for } \lambda\text{-a.e. } x \in \mathbb{R}^d.$$

Proof. What we have proved is $\lim_{r \to 0} \frac{1}{\lambda(B_r(x))} \int_{B_r(x)} (f(y) - f(x)) \, d\lambda(y) = 0$ for a.e. $x \in \mathbb{R}^d$. This doesn't help.

However, for each constant $c \in \mathbb{R}$, $\forall^\lambda x \in \mathbb{R}^d$, we have: $\lim_{r \to 0} \frac{1}{\lambda(B_r(x))} \int_{B_r(x)} |f - c| \, d\lambda = |f(x) - c|$.
In particular, we have this for all rational c, so intersecting ctbly many conull sets, we get a conull set $X \subseteq \mathbb{R}^d$ s.t. $\forall x \in X \ \forall q \in \mathbb{Q}$: $\lim_{r \to 0} \frac{1}{\lambda(B_r(x))} \int_{B_r(x)} |f - q| \, d\lambda = |f(x) - q|$.

Now fix $x \in X$ and put $c := f(x)$. Take $q \in \mathbb{Q}$, so $|f - c| \le |f - q| + |q - c|$, hence

$$\frac{1}{\lambda(B_r(x))} \int_{B_r(x)} |f - c| \, d\lambda \le \frac{1}{\lambda(B_r(x))} \int_{B_r(x)} |f - q| \, d\lambda + \frac{1}{\lambda(B_r(x))} \int_{B_r(x)} |q - c| \, d\lambda = \frac{1}{\lambda(B_r(x))} \int_{B_r(x)} |f - q| \, d\lambda + |q - c| \xrightarrow{r \to 0} |f(x) - q| + |q - c| = 2|q - c| \xrightarrow[q \to c]{} 0.$$ $\square$